

A-level

Mathematics

MFP4 – Further Pure 4 Mark scheme

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Mark schemes are prepared by the Lead Assessment Writer and considered, together with the relevant questions, by a panel of subject teachers. This mark scheme includes any amendments made at the standardisation events which all associates participate in and is the scheme which was used by them in this examination. The standardisation process ensures that the mark scheme covers the students' responses to questions and that every associate understands and applies it in the same correct way. As preparation for standardisation each associate analyses a number of students' scripts. Alternative answers not already covered by the mark scheme are discussed and legislated for. If, after the standardisation process, associates encounter unusual answers which have not been raised they are required to refer these to the Lead Assessment Writer.

It must be stressed that a mark scheme is a working document, in many cases further developed and expanded on the basis of students' reactions to a particular paper. Assumptions about future mark schemes on the basis of one year's document should be avoided; whilst the guiding principles of assessment remain constant, details will change, depending on the content of a particular examination paper.

Further copies of this mark scheme are available from aga.org.uk

Key to mark scheme abbreviations

M	mark is for method
m or dM	mark is dependent on one or more M marks and is for method
Α	mark is dependent on M or m marks and is for accuracy
В	mark is independent of M or m marks and is for method and
	accuracy
E	mark is for explanation
√or ft or F	follow through from previous incorrect result
CAO	correct answer only
CSO	correct solution only
AWFW	anything which falls within
AWRT	anything which rounds to
ACF	any correct form
AG	answer given
SC	special case
OE	or equivalent
A2,1	2 or 1 (or 0) accuracy marks
−x EE	deduct x marks for each error
NMS	no method shown
PI	possibly implied
SCA	substantially correct approach
С	candidate
sf	significant figure(s)
dp	decimal place(s)

No Method Shown

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

Otherwise we require evidence of a correct method for any marks to be awarded.

Q1	Solution	Mark	Total	Comment
(a)	$\left(\overrightarrow{AB} = \right) \begin{pmatrix} 2\\4\\-3 \end{pmatrix}$			
	$\left(\overline{AC} = \right) \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}$	B1		Either \overrightarrow{AB} or \overrightarrow{AC} correct
	$ \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & 2 & 1 \\ \mathbf{j} & 4 & 1 \\ \mathbf{k} & -3 & -4 \end{vmatrix} = \begin{pmatrix} -13 \\ 5 \\ -2 \end{pmatrix}$	M1 A1	3	Vector product – two components correct (unsimplified) CSO
(b)	Area = $\frac{1}{2}\sqrt{13^2 + 5^2 + 2^2}$	M1		Correct use of formula for area of a triangle with their vector
	$=\frac{3}{2}\sqrt{22}$	A1	2	Accept any exact equivalents eg $\frac{1}{2}\sqrt{198}$ CAO
	Total		5	

Q2	Solution	Mark	Total	Comment
	$\det (\mathbf{AB})^{-1} = -16$	B1		Correct evaluation of $\det(\mathbf{AB})^{-1}$ or
				$\det\left(\mathbf{A}\mathbf{B}\right) = -\frac{1}{16}$
	Either use of $\det \mathbf{A} \times \det \mathbf{B} = \det (\mathbf{AB})$ or			
	$\det \mathbf{A} \times \det \mathbf{B} = \det (\mathbf{A}\mathbf{B}) \text{ or}$ $\det \mathbf{B}^{-1} \times \det \mathbf{A}^{-1} = \det (\mathbf{A}\mathbf{B})^{-1}$			
	$\frac{1}{2} \times \det \mathbf{B}^{-1} = -16$	M1		Correct use of determinant rules and substituting their values
	$\det \mathbf{B} = -\frac{1}{32}$	A1	3	CAO
	Total		3	

Q3	Solution	Mark	Total	Comment
(a)(i)	$\begin{vmatrix} r_1 \text{ replaced by } r_1 - r_2 \text{ gives} \\ a - 4 & -12 + 3a - a + 4 \\ 4 & -3a & a - 3 \\ -3 & 4a & a + 4 \end{vmatrix}$	M1		Correct use of a row operation with at most one error in one term.
	$\begin{vmatrix} 1 & 3 & -1 \\ 4 & -3a & a-3 \\ -3 & 4a & a+4 \end{vmatrix}$	A1	2	Or r_2 replaced by $r_2 - r_1$ give $ \begin{vmatrix} a & -12 & 1 \\ -1 & -3 & 1 \\ -3 & 4a & a+4 \end{vmatrix} $ CAO
(ii)	c_3 replaced by $c_3 + c_1$ gives			
	$\begin{vmatrix} (a-4) & 1 & 3 & 0 \\ 4 & -3a & a+1 \\ -3 & 4a & a+1 \end{vmatrix}$	M1		Combining rows or columns sensibly working towards a second linear factor, to a point where a factor can be extracted.
	$\begin{vmatrix} (a-4)(a+1) & 1 & 3 & 0 \\ 4 & -3a & 1 \\ -3 & 4a & 1 \end{vmatrix}$	A1		Correct extraction of second linear factor and resulting determinant
	-7(a-4)(a+1)(a+3)	m1 A1	4	Correct expansion of their resulting determinant to find final factor Fully correct – must pull out the "7" for final A1. CSO
(b)	-7(a-4)(a+1)(a+3) = 0	M1		Set their answer to (a)(ii)equal to 0
	a = 4, -3, -1	A1	2	All three values obtained CSO – must have scored 4 marks in (a)(ii) SC – if 3 in (a)(ii) due to "7" not pulled out can get M1A1 .
(a)(ii)	ALTERNATIVE			
	By direct expansion $ \Delta = \begin{vmatrix} a & -12 & 1 \\ 4 & -3a & a - 3 \\ -3 & 4a & a + 4 \end{vmatrix} $ $ = a \begin{vmatrix} -3a & a - 3 \\ 4a & a + 4 \end{vmatrix} + 12 \begin{vmatrix} 4 & a - 3 \\ -3 & a + 4 \end{vmatrix} + \begin{vmatrix} 4 & -3a \\ -3 & 4a \end{vmatrix} $ $ = -7a^{3} + 91a + 84 $ $ = -7(a - 4)(a^{2} + 4a + 3) $	(M1) (A1)		Correct cubic obtained Correct quadratic factor seen with one linear
	= -7(a-4)(a+3)(a+1)	(A1) (A1) (A1)	(4)	factor identified Two linear factors correct Fully correct CSO
	Total		8	

Q4	Solution	Mark	Total	Comment
(a)	5 2 4 7 4 6 6 k-2 k			
	$ = 5 \begin{vmatrix} 4 & 6 \\ k - 2 & k \end{vmatrix} - 7 \begin{vmatrix} 2 & 4 \\ k - 2 & k \end{vmatrix} + 6 \begin{vmatrix} 2 & 4 \\ 4 & 6 \end{vmatrix} $	M1		Correct row or column expansion
	=5(-2k+12)-7(-2k+8)+6(-4)	m1		Correct 2 by 2 determinant expansion
	=4k-20			
	$4k - 20 \neq 0$	A1		Fully correct linear expression and correct conclusion – AG – CSO
	$k \neq 5$	111	3	conclusion 110 CSO
	ALTERNATIVE $\begin{vmatrix} 5 & 2 & 4 \\ 7 & 4 & 6 \\ 6 & 3 & 5 \end{vmatrix}$ Then row 1 + row 2 = 2 × row 3 so det $\mathbf{A} = 0$ since \mathbf{A} is non-singular then $k \neq 5$	(SC1)	(1)	Correct substitution of $k = 5$ and an attempt at row operations or evaluation of determinant Correct justification/evaluation of det $\mathbf{A} = 0$ Fully correct conclusion
(b)	$\begin{bmatrix} 12-2k & 36-7k & 7k-38 \\ 2k-8 & 5k-24 & 22-5k \\ -4 & -2 & 6 \end{bmatrix}$	M1 A(2,1,0)		M1 cofactor matrix – one full row or column correct A1 – at least six entries correct A2 – all entries correct
	$\mathbf{A}^{-1} = \frac{1}{4k - 20} \begin{bmatrix} 12 - 2k & 2k - 8 & -4 \\ 36 - 7k & 5k - 24 & -2 \\ 7k - 38 & 22 - 5k & 6 \end{bmatrix}$	m1 A1	5	Divide by their determinant and transpose their matrix – must have first M1 CAO

(c)	Comparing gives $k = -1$, hence $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-24} \begin{bmatrix} 14 & -10 & -4 \\ 43 & -29 & -2 \\ -45 & 27 & 6 \end{bmatrix} \begin{bmatrix} a \\ -a \\ -5a \end{bmatrix}$			Must be evidence of using their A ⁻¹ from part (b)
	$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-24} \begin{bmatrix} 14a + 10a + 20a \\ 43a + 29a + 10a \\ -45a - 27a - 30a \end{bmatrix}$	M1 A1F		M1 – Correct identification of $k = -1$ and attempt at \mathbf{A}^{-1} \mathbf{v} with at least one of their components correct – can be unsimplified, but with no k 's. A1F Two of their components correct – can be unsimplified. Follow through their \mathbf{A}^{-1} (Condone wrong or missing $1/ \mathbf{A} $ for M1A1F)
	$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-24} \begin{bmatrix} 44a \\ 82a \\ -102a \end{bmatrix}$	A1		All three components correct – terms collected but need not be fully cancelled
	$x = -\frac{11a}{6}$ $y = -\frac{41a}{12}$			
	$z = \frac{17a}{4}$	A1	4	Correct and fully simplified, with variables clearly identified – CSO
				Any method with does not use A^{-1} v scores zero marks $NMS = 0$ marks
	Total		12	

Q5	Solution	Mark	Total	Comment
(a)	$\begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} \begin{bmatrix} x \end{bmatrix}$			
	$\left \begin{array}{ccc c} 0 & 5 & -1 \end{array} \right \left \begin{array}{ccc c} y & \end{array} \right $			
	$\begin{bmatrix} 2 & p & 1 \end{bmatrix} \begin{bmatrix} z \end{bmatrix} \begin{bmatrix} z \end{bmatrix}$			
	Hence $(1) x + y + 2z = 2x$			
	(2) 5y - z = 2y	M1		Use of $\mathbf{M}\mathbf{v} = 2\mathbf{v}$ to obtain three correct equations
	$(3) \ 2x + py + z = 2z$			
	(2) simplifies as $z = 3y$			
	Using this with (1) gives $x = 7y$	A1		Correctly obtaining each variable in terms of a single variable/parameter
				single variable/parameter
	Substituting in (3) gives $14y + py - 3y = 0$	m1		Substituting in final equation to obtain an
				equation in p and one other variable
	Hence $p = -11$	A1	4	CSO
	ALTERNATIVE			
	Substituting $\lambda = 2$			
		(M1)		Fully some at constitute at in
	$\begin{bmatrix} 0 & 3 & -1 \\ 2 & p & -1 \end{bmatrix} = 0$	(M1)		Fully correct equation with $\lambda = 2$ substituted in.
	2 p-1			
	1 3 -1 2 2			
	$\begin{vmatrix} -1 \begin{vmatrix} 3 & -1 \\ p & -1 \end{vmatrix} + 2 \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = 0$	(A1)		Correct expansion by row or column of 3 by 3 determinant
	и 2 14 0	(m1)		
	Hence $3 - p - 14 = 0$	(m1)		Correct expansion of 2 by 2 determinants to form linear equation
	giving $p = -11$	(A1)	(4)	Correct value found - CSO
	ı	l 		
(b)	$\begin{vmatrix} 1 - \lambda & 1 & 2 \\ 0 & 5 - \lambda & -1 \\ 2 & -11 & 1 - \lambda \end{vmatrix} (=0)$			
	$0 5-\lambda -1 (=0)$			
	$\begin{vmatrix} 2 & -11 & 1-\lambda \end{vmatrix}$			
	$ 5-\lambda -1 $ $ 1 2 $	3.41		
	$(1-\lambda)\begin{vmatrix} 5-\lambda & -1 \\ -11 & 1-\lambda \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ 5-\lambda & -1 \end{vmatrix} (=0)$	M1		Correct row/column expansion of their $ \mathbf{M} - \lambda \mathbf{I} (= 0)$, may be in terms of p .
	$(1-\lambda)[(5-\lambda)(1-\lambda)-11]+2(-1-10+2\lambda)(=0)$	m1		Correct expansion of their 2 by 2 determinants – dependent on first M1 , may be in terms of <i>p</i> .
	$\lambda^3 - 7\lambda^2 - 4\lambda + 28 (=0)$	A1		· ·
	$(\lambda - 2)(\lambda + 2)(\lambda - 7) (= 0)$			Correct expanded characteristic equation
	$(\lambda - 2)(\lambda + 2)(\lambda - 1)(-0)$	A1		Correct factors
	$(\lambda_1 = 2), \lambda_2 = -2, \lambda_3 = 7$	A1	5	Correct eigenvalues obtained and identified

(c)	Any eigenvector found $\begin{bmatrix} 7 \\ 1 \\ 3 \end{bmatrix} \begin{bmatrix} -5 \\ 1 \\ 7 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -4 \end{bmatrix}$	M1		Obtaining a correct eigenvector
	Hence equation of line is $\mathbf{r} = \begin{bmatrix} 7 \\ 1 \\ 3 \end{bmatrix} t$	A1	2	Any fully correct vector format Must have " $\mathbf{r} = \cdots$ "
	Total		11	

Q6	Solution	Mark	Total	Comment
(a)	All points on the line remain in the same position (after transformation)	E1	1	Must explain what "invariant" means.
(b)	T maps (x, y) to (x, y) x = -2x + 2y + 4 y = 3x - y - 4	M1 A1		Replace either <i>x</i> ' by <i>x</i> or <i>y</i> ' by <i>y</i> . Replace both <i>x</i> ' & <i>y</i> ' by <i>x</i> & <i>y</i>
	Any correct simplified form e.g. $0 = 3x - 2y - 4$ (is the line of invariant points).	A1	3	Correct line identified – ACF (e.g. $y = \frac{3}{2}x - 2$)
(c)	Using $y = mx + c$ gives x' = -2x + 2(mx + c) + 4 y' = 3x - (mx + c) - 4	B1		Correct substitution of $y = mx + c$
	Then using $y' = mx' + c$ gives $3x - y - 4 = m(-2x + 2y + 4) + c$	M1		Substitution of $y' = mx' + c$
	$3x - (mx+c) - 4 = m[-2x + 2(mx+c) + 4] + c$ $0 = (2m^2 - m - 3)x + (2mc + 4m + 2c + 4)$	A1		Fully correct simplification – collecting appropriately
	0 = (2m-3)(m+1)x + 2(m+1)(c+2) Hence when $m = -1$, c can be any real value	m1		Factorising and solving to find m , clear reference to putting $m = -1$ into the other eq.
	So $y = -x + c$ are the infinite set of invariant lines [NB $m = \frac{3}{2}, c = -2$ gives line of invariant	A1	5	Fully correct conclusion – CSO
	points]		0	
	Total		9	

Q7	Solution	Mark	Total	Comment
(a)	Determinant of $\mathbf{M} = 8k^3$	B1	1	Correct evaluation of det M
(b)(i)	Volume scale factor = $\frac{0.75}{48} = \frac{1}{64}$ Hence $8k^3 = \frac{1}{64}$	M1		Forms a correct equation in <i>k</i> PI correct <i>k</i> or Scale Factor
	so $k = \frac{1}{8}$ and	A1		k value correct
	enlargement scale factor = $\frac{1}{4}$	A1	3	Scale factor correct
(ii)	Let \mathbf{N} = required matrix, hence $\mathbf{N} \begin{pmatrix} 2k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 2k \end{pmatrix} = \begin{pmatrix} -k\sqrt{3} & 0 & -k \\ 0 & 2k & 0 \\ k & 0 & -k\sqrt{3} \end{pmatrix}$	M1		Combines appropriate matrices to find required matrix associated with $T_2 - k$ value may be substituted.
	$\mathbf{N} = \begin{bmatrix} -\sqrt{3} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{-\sqrt{3}}{2} \end{bmatrix}$	A1		Correct matrix obtained.
	T_2 is a rotation about the y axis	M1		And no other transformation given.
	$\cos \theta = -\frac{\sqrt{3}}{2}$ and $\sin \theta = -\frac{1}{2}$	m1		PI by 210 ⁰ OE
	210^{0}	A1	5	Correct angle found – also accept –150 ^o (any equivalences accepted)
	Total		g	
	Total		9	

Q8	Solution	Mark	Total	Comment
(a)	$\mathbf{n} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$ $c = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = 5(3) + (2)(-1) + (-1)(2)$	M1		Correct use of scalar product with correct n and correct point to find a value of c
	$\mathbf{r} \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} = 11$	A1	2	Correct value of c and use of given format
(b)	$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix} \text{ or } \mathbf{v}_2 = \begin{bmatrix} -2 \\ -4 \\ 0 \end{bmatrix} \text{ or } \mathbf{v}_3 = \begin{bmatrix} -4 \\ -2 \\ 3 \end{bmatrix}$	B1		Any correct direction vector
	$\mathbf{n} = \begin{vmatrix} \mathbf{i} & 2 & -2 \\ \mathbf{j} & -2 & -4 \\ \mathbf{k} & -3 & 0 \end{vmatrix} = \begin{bmatrix} -12 \\ 6 \\ -12 \end{bmatrix}$	M1 A1		Using vector product of two correct direction vectors
	Hence $-12x + 6y - 12z = p$ p = -30	m1		Attempts to use their \mathbf{n} and correct point to find their value of p
	2x - y + 2z = 5	A1	5	d = 5 CSO
(c)	$\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = 6$	В1		Correct evaluation of scalar product
	$\sqrt{5^2 + 2^2 + 1^2} = \sqrt{30}$ $\sqrt{2^2 + 1^2 + 2^2} = 3$ $\cos \theta = \frac{6}{3\sqrt{30}}$ $(=\frac{6\sqrt{30}}{90}) = \frac{\sqrt{30}}{15}$	M1		Use of correct normals with $\cos \theta = \frac{\mathbf{a.b}}{ \mathbf{a} \mathbf{b} }$
	$(=\frac{6\sqrt{30}}{90}) = \frac{\sqrt{30}}{15}$	A1	3	AG – Be convinced

	ALTERNATIVE			
	$\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} \times \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \sqrt{234}$	(B1)		Correct evaluation of vector product. Must have modulus
	$\sin \theta = \frac{\sqrt{234}}{3\sqrt{30}}$	(M1)		Use of correct normals with $\sin \theta = \frac{ \mathbf{a} \times \mathbf{b} }{ \mathbf{a} \mathbf{b} }$
	$\cos^2 \theta = 1 - \sin^2 \theta = 1 - \frac{26}{30} = \frac{4}{30}$			
	$\therefore \cos \theta = \frac{\sqrt{30}}{15}$	(A1)	(3)	$\mathbf{AG - Must} \text{ use } \cos^2 \theta + \sin^2 \theta = 1$ else $\mathbf{A0}$
(d)(i)	$\begin{vmatrix} 5 & 2 & -1 \\ 2 & -1 & 2 \\ 4k - 8 & k + 1 & -4k^2 \end{vmatrix} = 0$			
	$\begin{vmatrix} (4k-8) \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - (k+1) \begin{vmatrix} 5 & -1 \\ 2 & 2 \end{vmatrix} - 4k^2 \begin{vmatrix} 5 & 2 \\ 2 & -1 \end{vmatrix} = 0$	M1		Correct expansion by row or column. (Substituting $k=1$ scores $M0$).
	$(4k-8)(3) - (k+1)(12) - 4k^{2}(-9) = 0$	A1		Correct expansion of 2 by 2 determinants
	$36k^2 - 36 = 0$ $36(k-1)(k+1) = 0$	m1		Valid method for solving a quadratic – dependent on first M1
	k = 1, -1	A1	4	Both values correct $-\mathbf{AG}$ for $k=1$
(ii)	$k=1$ results in the equation for Π_3 as $-4x+2y-4z=8$. Or valid method for comparing normals of Π_3 and Π_2	M1		Substituting $k = 1$ to get correct equation for Π_3 .
	Π_3 and Π_2 are parallel (Π_1 is not parallel).	A1		Must identify correct parallel planes.
	$k = -1$ results in the equation for Π_3 as $-12x - 4z = 8$.	M1		Substituting $k = -1$ to get correct equation for Π_3 . Must have scored 4 marks in (d)(i).
	Since planes have no common point . The three planes form a prism .	A1	4	Fully correct deduction
	Total		18	
	TOTAL		75	